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CALCULATION OF TEMPERATURE AND MOISTURE DISTRIBUTIONS DURING CONTACT DRYING OF A SHEET OF MOIST MATERIAL

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Abstract—Using equations derived by Luikov [7–9] as a starting point, the drying of a layer of moist material in contact with a hot plate is analysed. Makovozov [5, 6] analysed a special case, viz. that of a constant temperature at the hot plate; we pay special attention to (i) the case of a constant heat flux and (ii) the case with a heat flux exponentially decreasing with time. As Makovozov we assume that (i) the pressure gradient influences on the moisture movement are negligible and (ii) that the moisture transport takes place mainly as a result of temperature gradients and not in consequence of moisture potential gradients. We endeavoured to derive also a more general equation without the last mentioned restriction but encountered unsurmountable difficulties due to the asymmetry of the boundary conditions involved.

The influence of dimensionless parameters on the temperature and moisture potential distributions is illustrated by numerical examples. The results are compared with scarce experimental data in the References.

NOMENCLATURE

x, coordinate perpendicular to the surface [L];
t, temperature [°C];
θ, moisture potential [°M]‡;
τ, time [T];
a_q, thermal diffusivity coefficient [L^2T^{-1}];
a_m, diffusion coefficient of moisture in the material [L^2T^{-1}];
r, specific heat of evaporation [L^2T^{-2}];
δ, thermal gradient coefficient [$°C^{-1}$];
c_m, specific isothermal mass capacity of the material [$°M^{-1}$];
c_q, specific heat capacity of the material [$L^2T^{-2}\,°C^{-1}$];
α_q, heat-transfer coefficient [$mT^{-3}\,°C^{-1}$]§;
α_m, mass-transfer coefficient [$mL^{-2}T^{-1}\,°M^{-1}$];
λ_q, thermal conductivity [$mLT^{-3}\,°C^{-1}$];
λ_m, $= c_m a_m \rho_m$, moisture conductivity [$mL^{-1}T^{-1}\,°M^{-1}$];
ρ, density of the dry material [mL^{-3}];
ϕ_q, heat flux [mT^{-3}];
ϕ_m, mass flux [$mL^{-2}T^{-1}$];
$\bar{G}_i$, partial specific Gibbs free energy [L^2T^{-1}];
L_{qi}, L_{iq}, phenomenological coefficients for the fluxes ϕ_m, ϕ_q [$mT^{-1}L^{-1}\,°C^{-1}$, $mT^{-1}L^{-1}$];
L_{ii}, $= \lambda_m/(\partial\bar{G}_i/\partial\theta)_t$, phenomenological coefficient for the flux ϕ_m [mTL^{-3}];
D, thickness of the layer of moist material [L];
σ, relaxation time [T].

Dimensionless criteria

Fo, Fourier number, $= a_q\tau/D^2$;
Lu, Luikov number, $= a_m/a_q$;
Bi_m, Biot number for mass transfer, $= \alpha_q D/\lambda_q$;
Bi_q, Biot number for heat transfer, $= \alpha_m D/\lambda_m$;

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‡See Luikov [7–9], $c_m = (\partial u/\partial\theta)_T$ where u is moisture content in [m/m].

§ m means a mass unit.

Ko, Kossovitch number,
$= \dfrac{rc_m(\theta_0 - \theta_*)}{c_q(t_k - t_0)}$;

Pn, Posnov number,
$= \dfrac{\delta(t_k - t_0)}{c_m(\theta_0 - \theta_*)}$;

ε, phase change criterion [4]
$= \dfrac{\partial u_l/\partial \tau}{\partial u/\partial \tau}$;

T, dimensionless temperature,
$= \dfrac{t - t_0}{t_k - t_0}$;

Θ, dimensionless moisture transfer potential,
$= \dfrac{\theta_0 - \theta}{\theta_0 - \theta_*}$;

ξ, dimensionless temperature of the surroundings,
$= \dfrac{t_s - t_0}{t_k - t_0}$;

X, dimensionless coordinate, $= x/D$;

$\nu, \varphi, \Lambda, \Pi$, dimensionless groups respectively defined in equaticns (26) and (35);

$Ki_q(Fo)$, dimensionless heat flux,
$= \dfrac{D\phi_{q,k}(Fo)}{\lambda_q(t_s - t_0)}$;

$Ki_m(Fo)$, dimensionless mass flux,
$= \dfrac{D\phi_{m,k}(Fo)}{\lambda_m(\theta_0 - \theta_*)}$.

Subscripts

k, at the surface of the hot plate;

0, initial;

s, surroundings;

$*$, in equilibrium with surrounding air;

$\langle\ \rangle$, averaged over characteristic surface;

l, in liquid state.

1. INTRODUCTION

IN THE present paper we analyse the contact drying of a moist porous sheet on a hot plate under such conditions that no boiling of the water is possible. Heat flows through the sheet from bottom to top (Fig. 1). The temperature at the surface may be either higher or lower than

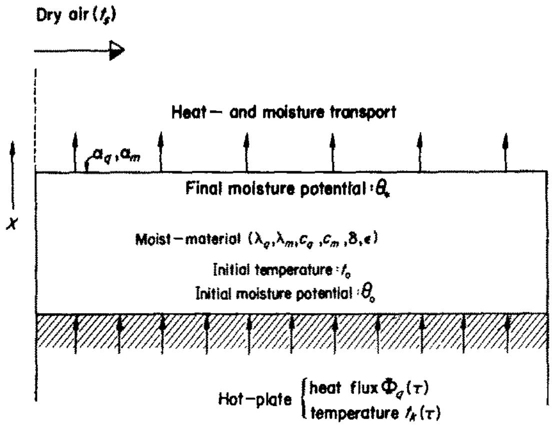

FIG. 1. Schematic picture of the contact drying process.

that of the surrounding air, so that the sheet may loose or gain heat by free or forced convection. It is assumed that the constitutive equations for heat and mass flux of thermodynamically irreversible processes hold (de Groot and Mazur [1], Chapter XV). According to these equations there is a liner relation between the fluxes (averaged over a representative surface element, sufficiently large to cover all occurring pore sizes) and the gradients in temperature and the temperature independent part of the partial specific Gibbs free energy (chemical potential). This is only true with one dimensional fluxes [2]. The gradients are defined as the differences in the parameters at two sides of a characteristic volume element divided by a characteristic length (see Whitaker [3]). The characteristic length must be long enough to cover all occurring sequences of pore sizes. With respect to an inertial frame of reference one obtains the following equations for the average fluxes:

$$\langle\phi_q\rangle = -\lambda_q \frac{\partial t}{\partial x} - L_{iq}\left(\frac{\partial \bar{G}_i}{\partial x}\right)_T$$
$$\langle\phi_m\rangle = -L_{qi} \frac{\partial t}{\partial x} - L_{ii}\left(\frac{\partial \bar{G}_i}{\partial x}\right)_T. \tag{1}$$

These equations are valid, or at least a good approximation, if the following conditions are satisfied:

1. During the process no deformation of the solid phase occurs (e.g. shrinkage phenomena are excluded).
2. The characteristic length of the porous medium must be some orders of magnitude smaller than the chararacteristic length of the spatial derivatives of the intensive parameters. Whitaker [3] gives a precise definition of these two characteristic lengths.
3. If there exist appreciable fluctuations in the intensive parameters inside a characteristic volume element (characteristic surface times the characteristic length) the averaging procedure is no longer admissible. Luikov [4, pp. 24, 26] assumes thermodynamic equilibrium between all three phases inside a volume element. In the thermodynamics of irreversible processes for a single phase a postulate is required which states that thermodynamic equilibrium exists at each point in the phase. If one considers a porous medium evidently one additional postulate has to be introduced to obtain the equations (1), namely that also thermodynamic equilibrium exists between the phases present inside a characteristic volume element. The main reason for the necessity of this postulate is the otherwise not uniquely defined temperature. If the experimental situation is such that this additional postulate is not warranted the equations (1) cannot be used.
4. The gradients in temperature and the chemical potential must be one dimensional. The necessity of this condition appears in the averaging of the entropy production over the characteristic volume element.
5. The rate of work done by viscous forces to push the fluid through the porous structure inside a volume element is negligible.

Combination of the equations (1) with the appropriate equations of change for each phase and subsequent summation over the phases present (the Gibbs free energy is assumed to be only a function of temperature and moisture transfer potential θ, gravity forces are neglected), led Luikov to the following set of differential equations (we give the one dimensional form):

$$\rho c_q \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial X}\left(\lambda_q \frac{\partial t}{\partial x}\right) + \varepsilon r c_m \rho \frac{\partial \theta}{\partial \tau} \tag{2}$$

$$\rho c_m \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda_m \frac{\partial \theta}{\partial x} + \partial \frac{\partial t}{\partial x}\right). \tag{3}$$

These equations govern the heat and mass transport inside the layer of moist porous material.

If λ_m, λ_q and δ are constant the following set of equations remains after introduction of dimensionless parameters:

$$\frac{\partial T}{\partial Fo} = \frac{\partial^2 T}{\partial X^2} - \varepsilon Ko \frac{\partial \Theta}{\partial Fo} \tag{4}$$

$$\frac{\partial \Theta}{\partial Fo} = Lu \frac{\partial^2 \Theta}{\partial X^2} - Lu\, Pn \frac{\partial^2 T}{\partial X^2}. \tag{5}$$

In practice λ_m, λ_q and δ will be functions of the moisture transfer potential and the temperature. For a zonal calculation one can consider them to be constant, see Luikov [4], p. 33.

Makovozov [5, 6] analysed the process of contact drying using these differential equations with the simplifying assumption that the moisture movement takes place mainly due to temperature gradients so that the influence of moisture potential gradients are negligible in comparison.

In section 2.1 we endeavoured to derive a solution of the complete set of equations (4) and (5) when the temperature of the hot plate is constant; Section 2.2 gives the analysis (with Makovozov's simplification) when the heat flux rather than the temperature is a given function of time.

2. DERIVATION OF EQUATIONS FOR TEMPERATURE AND MOISTURE DISTRIBUTION

2.1. *General solutions*

The equations (4) and (5) must be solved with the following boundary conditions:

Initial conditions:

$$T(X,0) = 0. \qquad \Theta(X,0) = 0. \tag{6}$$

Heat balance at the free surface:

$$\frac{\partial}{\partial X} T(1, Fo) - Bi_q[\xi - T(1, Fo)] - (1 - \varepsilon) \times Ko\, Lu\, Bi_m[1 - \Theta(1, Fo)] = 0. \tag{7}$$

Mass balance at the free surface:

$$-\frac{\partial}{\partial X}\Theta(1, Fo) + Pn\frac{\partial}{\partial X}T(1, Fo) + Bi_m[1 - \Theta(1, Fo)] = 0. \tag{8}$$

Mass balance at the surface of the hot plate:

$$\frac{\partial}{\partial X}\Theta(0, Fo) - Pn\frac{\partial}{\partial X}T(0, Fo) + Ki_m(Fo) = 0. \tag{9}$$

The temperature of the hot plate is constant (i), or the heat flux is given (ii):

$$\left.\begin{array}{l} \text{(i)} \quad T(0, Fo) = 1 \\ \text{(ii)} \quad \dfrac{\partial}{\partial X}T(0, Fo) + Ki_q(Fo) \\ \qquad - (1 - \varepsilon)\, KoLu\, Ki_m(Fo) = 0. \end{array}\right\} \tag{10}$$

This is the set of boundary conditions Makovozov uses, if one puts $Ki_m(Fo) = 0$ in equation (9) (this means that no mass transfer takes place at the surface of the hot plate) and if we choose (10.i). We assumed linear heat and mass transfer at the free surface of the material. It is convenient to solve this set of equations by means of the Laplace transformation [4, 10–12]; we define the transformation as:

$$\mathscr{L}\{\phi(X, Fo)\} \equiv \overline{\phi}(X, s) = \int_0^\infty \phi(X, Fo)\exp(-sFo)\,\mathrm{d}Fo. \tag{11}$$

Applying this transformation on (4–10) gives:

$$s\overline{T}(X, s) = \frac{\mathrm{d}^2}{\mathrm{d}X^2}\overline{T}(X, s) - \varepsilon\, Ko\, s\,\overline{\Theta}(X, s) \tag{12}$$

$$s\,\Theta(X, s) = Lu\frac{\mathrm{d}^2}{\mathrm{d}X^2}\overline{\Theta}(X, s) - Lu\, Pn\frac{\mathrm{d}^2}{\mathrm{d}X^2}\overline{T}(X, s) \tag{13}$$

and the boundary conditions become:

$$\frac{\mathrm{d}}{\mathrm{d}X}\overline{T}(1, s) - Bi_q\left[\frac{\xi}{s} - \overline{T}(1, s)\right] + (1 - \varepsilon)\, Lu\, Ko\, Bi_m\left[\frac{1}{s} - \overline{\Theta}(1, s)\right] = 0 \tag{14}$$

$$Pn\frac{\mathrm{d}}{\mathrm{d}X}\overline{T}(1, s) - \frac{\mathrm{d}}{\mathrm{d}X}\overline{\Theta}(1, s) + Bi_m\left[\frac{1}{s} - \overline{\Theta}(1, s)\right] = 0 \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\overline{\Theta}(0, s) - Pn\frac{\mathrm{d}}{\mathrm{d}X}\overline{T}(0, s) + \overline{Ki}_m(s) = 0 \tag{16}$$

$$\overline{T}(0, s) = \frac{1}{s} \tag{17.i}$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\overline{T}(0, s) + \overline{Ki}_q(s) - (1 - \varepsilon)\, Ko\, Lu\,\overline{Ki}_m(s) = 0. \tag{17.ii}$$

The solutions of (12) and (13) are [4]:

$$\overline{T}(X, s) = A_1\exp(X\Pi_1\sqrt{s}) + A_2\exp(-X\Pi_1\sqrt{s}) + A_3\exp(X\Pi_2\sqrt{s}) + A_4\exp(-X\Pi_2\sqrt{s}) \tag{18}$$

$$\overline{\Theta}(X, s) = \frac{(1 - \Pi_1^2)}{\varepsilon Ko}[A_1\exp(X\Pi_1\sqrt{s}) + A_2\exp(-\Pi_1 X\sqrt{s})] + \frac{(1 - \Pi_2^2)}{\varepsilon Ko}[A_3\exp(X\Pi_2\sqrt{s}) + A_4\exp(-X\Pi_2\sqrt{s})]. \tag{19}$$

The constants A_1–A_4 can in principle be calculated from (14–17). The Π_1 and Π_2 in (18) and (19) are given by:

$$\Pi^2_{i \atop i=1,2} = \frac{1}{2}\left\{1 + \varepsilon\, Ko\, Pn + \frac{1}{Lu} - (-1)^i \sqrt{\left[\left(1 + \varepsilon\, Ko\, Pn + \frac{1}{Lu}\right)^2 - \frac{4}{Lu}\right]}\right\}. \tag{20}$$

The calculation of the constants $A_1 - A_4$ from (15, 14, 16) with $\overline{Ki}_m(s) = 0$ and (17.i) gives the following set of linear equations:

$$\left.\begin{aligned}
&A_1 + A_2 + A_3 + A_4 = \frac{1}{s}\\
&N_1(\sqrt{s})A_1 + N_2(\sqrt{s})A_2 - N_1(\sqrt{s})A_3 - N_2(\sqrt{s})A_4 = 0\\
&M_1A_1\exp(\Pi_1\sqrt{s}) - R_1A_2\exp(-\Pi_1\sqrt{s}) + M_2A_3\exp(\Pi_2\sqrt{s}) - R_2A_4\exp(-\Pi_2\sqrt{s})\\
&\qquad = -\frac{Bi_m}{s}\\
&[\Pi_1(\sqrt{s}) + Q_1]A_1\exp(\Pi_1\sqrt{s}) - [\Pi_1(\sqrt{s}) - Q_1]A_2\exp(-\Pi_1\sqrt{s})\\
&\quad + [\Pi_2(\sqrt{s}) + Q_2]A_3\exp(\Pi_2\sqrt{s}) - [\Pi_2(\sqrt{s}) - Q_2]A_4\exp(-\Pi_2\sqrt{s})\\
&\qquad = \frac{1}{s}[Bi_q\xi - (1-\varepsilon)\, Ko\, Lu\, Bi_m]
\end{aligned}\right\} \tag{21}$$

where:

$$\left.\begin{aligned}
N_i &= -Pn\,\Pi_i + \frac{\Pi_i(1-\Pi_i^2)}{\varepsilon\, Ko}\\
M_i &= N_i - \frac{Bi_m(1-\Pi_i^2)}{\varepsilon\, Ko}\\
R_i &= N_i + \frac{Bi_m(1-\Pi_i^2)}{\varepsilon\, Ko}\\
Q_i &= Bi_q - \frac{(1-\varepsilon)}{\varepsilon}\, Lu\, Bi_m(1-\Pi_i^2).
\end{aligned}\right\} \tag{22}$$

The system of equations (21, 22) can in principle be solved by the well known Cramers rule [13], but it will be very difficult to perform the inversion of the Laplace transformed potentials. It is clear that these complications arise from the asymmetrical boundary conditions to the problem.

2.2. *Simplified models*

Makovozov proposed to simplify the system of equations by putting

$$Lu\frac{\partial^2}{\partial X^2}\Theta \sim 0.$$

Physically this means that the moisture movement under influence of moisture potential gradients is negligible. In the first periods of the drying process this condition will be certainly fulfilled. One could say that if the Posnow number has a relatively high value this approximation seems reasonable. Further discussion is postponed to Section 3.

Accepting this simplification the equations (12–17) become [equation (14) and (15) are combined to give equation (25)]:

$$s\overline{T}(X,s) = \nu\frac{d^2}{dX^2}\overline{T}(X,s) \tag{23}$$

$$s\overline{\Theta}(X,s) = -Lu\,Pn\frac{d^2}{dX^2}\overline{T}(X,s) \tag{24}$$

$$\frac{d}{dX}\overline{T}(1,s) - \varphi Bi_q\left[\overline{T}(1,s) - \frac{\xi}{s}\right] - \varphi(1-\varepsilon)\, Ko\, Lu\frac{d}{dX}\overline{\Theta}(1,s) = 0 \tag{25}$$

where $\nu = 1 + \varepsilon\, Ko\, Lu\, Pn$

and $\varphi = [1 - (1-\varepsilon)\, Ko\, Lu\, Pn]^{-1}$ (26)

$$\left.\begin{aligned} &\text{(i)}\quad \frac{\mathrm{d}}{\mathrm{d}X}\bar{\Theta}(0,s) - Pn\frac{\mathrm{d}}{\mathrm{d}X}\bar{T}(0,s) = 0 \\ &\text{(ii)}\quad \bar{T}(0,s) = \frac{1}{s} \end{aligned}\right\} \text{from (16), with } \overline{Ki}_m(s) = 0, \text{ and (17.i)} \tag{27}$$

$$\left.\begin{aligned} &\text{(i)}\quad \frac{\mathrm{d}}{\mathrm{d}X}\bar{\Theta}(0,s) - Pn\frac{\mathrm{d}}{\mathrm{d}X}\bar{T}(0,s) + \overline{Ki}_m(s) = 0 \\ &\text{(ii)}\quad \frac{\mathrm{d}}{\mathrm{d}X}\bar{T}(0,s) + \overline{Ki}_q(s) - (1-\varepsilon)\,Ko\,Lu\,\overline{Ki}_m(s) = 0 \end{aligned}\right\} \text{from (16) and (17.ii).} \tag{28}$$

The solution of (23) and (24) can be written as

$$\bar{T}(X,s) = A\cosh\left[X\sqrt{\left(\frac{s}{v}\right)}\right] + B\sinh\left[X\sqrt{\left(\frac{s}{v}\right)}\right] \tag{29}$$

$$\bar{\Theta}(X,s) = \frac{Lu\,Pn}{v}\left\{A\cosh\left[X\sqrt{\left(\frac{s}{v}\right)}\right] + B\sinh\left[X\sqrt{\left(\frac{s}{v}\right)}\right]\right\} \tag{30}$$

with A and B constants to be determined from the boundary conditions. Makovozov choose to satisfy the conditions (25) and (27.ii), which means physically that the temperature of the hot plate is a constant [5]. A and B then become:

$$A = \frac{1}{s} \tag{31}$$

$$B = -\frac{\sinh\left[\sqrt{\left(\frac{s}{v}\right)}\right] + \varphi\,Bi_q\left[\sqrt{\left(\frac{v}{s}\right)}\right]\left\{\cosh\left[\sqrt{\left(\frac{s}{v}\right)}\right] - 1\right\} - \varphi(1-\varepsilon)\,Ko\,Lu^2\,Pn\sinh\left[\sqrt{\left(\frac{s}{v}\right)}\right]}{\cosh\left[\sqrt{\left(\frac{s}{v}\right)}\right] + \varphi\,Bi_q\left[\sqrt{\left(\frac{v}{s}\right)}\right]\sinh\left[\sqrt{\left(\frac{s}{v}\right)}\right] - \varphi(1-\varepsilon)\,Ko\,Lu^2\,Pn\cosh\left[\sqrt{\left(\frac{s}{v}\right)}\right]} \tag{32}$$

This value for B is not the same as Makovozov obtains, the $\varphi(1-\varepsilon)\,Ko\,Lu^2$ . . . terms are missing because he puts the last term in (25) equal to zero. If we substitute (31) and (32) into the boundary condition (27.i) we see that they are not compatible with it, equation (27.i) requiring B to be zero. In fact one could say that the combination of (27.i) and (27.ii) as boundary conditions at the hot plate surface are not satisfactory in the simplified model.†

We now construct a solution with the aid of the boundary conditions (25), (28.i) and (28.ii). Physically this means that we accept the heat flux from the hot plate as a given function of time, rather than the temperature. The dimensionless parameters with $(t_k - t_0)$ now have $(t_s - t_0)$ instead, ξ becomes unity. Combining (28.i) and (28.ii) gives (eliminating $\overline{Ki}_m(s)$):

$$\frac{\mathrm{d}}{\mathrm{d}X}\bar{T}(0,s) + \varphi\overline{Ki}_q(s) + (1-\varepsilon)\,Ko\,Lu\,\varphi\frac{\mathrm{d}}{\mathrm{d}X}\bar{\Theta}(0,s) = 0. \tag{33}$$

Substituting from (29) and (30) in (33) gives the value of B:

$$B = -A\left(\sqrt{\frac{v}{s}}\right)\overline{Ki}_q(s). \tag{34}$$

† The reason for this incompatibility is the fact that (27.i) and (27.ii) cannot be combined to give one single boundary condition.

Here we have defined new complex dimensionless criteria Π, Λ as follows:

$$\Lambda = \frac{\varphi}{1 + \Pi} = \frac{\varphi}{1 + [Ko\, Lu^2\, Pn\, \varphi(1 - \varepsilon)]/\nu}. \tag{35}$$

From (29), (30) and (25) we obtain from (34) the value of A:

$$A = \frac{\varphi\, Bi_q + s\,\Lambda\,\overline{Ki}_q(s)\left[(\Pi + 1)\cosh\sqrt{\left(\frac{s}{\nu}\right)} + \varphi\, Bi_q \sqrt{\left(\frac{\nu}{s}\right)}\sinh\sqrt{\left(\frac{s}{\nu}\right)}\right]}{s\left[\varphi\, Bi_q \cosh\sqrt{\left(\frac{s}{\nu}\right)} + (\Pi + 1)\sqrt{\left(\frac{s}{\nu}\right)}.\sinh\sqrt{\left(\frac{s}{\nu}\right)}\right]}, \tag{36}$$

Substituting (34) and (36) in (29) we obtain:

$$\bar{T}(X, s) = \frac{\varphi\, Bi_q \cosh\left[\sqrt{\left(\frac{s}{\nu}\right)}.X\right]}{s\left[\varphi\, Bi_q \cosh\sqrt{\left(\frac{s}{\nu}\right)} + (1 + \Pi)\sqrt{\left(\frac{s}{\nu}\right)}\sinh\sqrt{\left(\frac{s}{\nu}\right)}\right]}$$
$$+ \Lambda\overline{Ki}_q(s)\frac{\left\{\varphi\, Bi_q \sqrt{\left(\frac{\nu}{s}\right)}.\sinh\left[\sqrt{\left(\frac{s}{\nu}\right)}.(1 - X)\right] + (1 + \Pi)\cosh\left[\sqrt{\left(\frac{s}{\nu}\right)}.(1 - X)\right]\right\}}{\varphi\, Bi_q \cosh\sqrt{\left(\frac{s}{\nu}\right)} + (1 + \Pi)\sqrt{\left(\frac{s}{\nu}\right)}.\sinh\sqrt{\left(\frac{s}{\nu}\right)}} \tag{37}$$

We invert this equation with the use of the Heaviside expansion theorem, the convolution theorem and the linearity properties of the Laplace transformation [10, 12], details are given in Appendix A. The result is:

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} A_n \cos(\mu_n x)\exp(-\mu_n^2 \nu\, Fo)$$
$$+ \nu\Lambda \sum_{n=1}^{\infty} \frac{A_n \mu_n \cos(\mu_n x)}{\sin\mu_n}\int_0^{Fo} Ki_q(\tilde{\tau})\exp[-\mu_n^2 \nu(Fo - \tilde{\tau})]\,d\tilde{\tau} \tag{38}$$

where $\tilde{\tau}$ is a dimensionless time dummy variable.

The μ_n are the positive roots of the characteristic equation

$$\mu\tan\mu = \frac{\varphi\, Bi_q}{1 + \Pi} = \Lambda\, Bi_q \tag{39}$$

and the A_n are given by:

$$A_n = \frac{2\sin\mu_n}{\mu_n + \sin\mu_n\cos\mu_n}. \tag{40}$$

Values for A_n are tabulated by Luikov [4], p. 158.

For the moisture potential distribution one obtains:

$$\Theta(X, Fo) = \frac{Lu\, Pn}{\nu}\sum_{n=1}^{\infty} A_n \cos(\mu_n x)[1 - \exp(-\mu_n^2 \nu\, Fo)]$$
$$+ Lu\, Pn\, \Lambda \sum_{n=1}^{\infty} \frac{A_n \mu_n \cos(\mu_n X)}{\sin\mu_n}\int_0^{Fo} Ki_q(\tilde{\tau})\exp[-\mu_n^2 \nu(Fo - \tilde{\tau})]\,d\tilde{\tau}. \tag{41}$$

The integral in (38) and (41) can be determined by a numerical procedure if $Ki_q(Fo)$ is known by experiment, see Appendix B. *If Ki_q is a constant* then (38) and (41) become:

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} A_n \cos(\mu_n X) \exp(-\mu_n^2 \nu\, Fo) + \Lambda\, Ki_q \sum_{n=1}^{\infty} \frac{A_n \cos(\mu_n X)}{\mu_n \sin \mu_n}[1 - \exp(-\mu_n^2 \nu\, Fo)] \tag{42}$$

$$\Theta(X, Fo) = \frac{Lu\, Pn}{\nu} \sum_{n=1}^{\infty} A_n \cos(\mu_n X)[1 - \exp(-\mu_n^2 \nu\, Fo)] + \frac{Lu\, Pn\, \Lambda\, Ki_q}{\nu} \sum_{n=1}^{\infty} \frac{A_n \cos(\mu_n X)}{\mu_n \sin \mu_n}[1 - \exp(-\mu_n^2 \nu\, Fo)]. \tag{43}$$

If $Ki_q(Fo)$ is an exponentially decreasing heat flux with relaxation time σ:

$$Ki_q(Fo) = Ki_q^0 \exp\left(-\frac{Fo}{\sigma}\right) \tag{44}$$

we obtain for the temperature and moisture potential distribution:

$$T(X, Fo) = 1 - \sum_{n=1}^{\infty} A_n \cos(\mu_n X) \exp(-\mu_n^2 \nu\, Fo) + \nu \Lambda\, Ki_q^0 \sigma \sum_{n=1}^{\infty} \frac{A_n \mu_n \cos(\mu_n X)}{\sin \mu_n (\sigma \mu_n^2 \nu - 1)} \left[\exp\left(-\frac{Fo}{\sigma}\right) - \exp(-\mu_n^2 \nu\, Fo)\right] \tag{45}$$

and

$$\Theta(X, Fo) = \frac{Lu\, Pn}{\nu} \sum_{n=1}^{\infty} A_n \cos(\mu_n X)[1 - \exp(-\mu_n^2 \nu\, Fo)] + \sigma Lu\, Pn\, \Lambda\, Ki_q^0 \sum_{n=1}^{\infty} \frac{A_n \mu_n \cos(\mu_n X)}{\sin \mu_n (\sigma \mu_n^2 \nu - 1)} \left[\exp\left(-\frac{Fo}{\sigma}\right) - \exp(-\mu_n^2 \nu\, Fo)\right]. \tag{46}$$

3. DISCUSSION OF THE RESULTS AND NUMERICAL EXAMPLES

Inspection of equation (38) and (41) reveals some interesting points. The first two terms of equation (38) describe the penetration of heat into the material from the free surface. The last term describes the temperature rise of the material by the heat flux from the hot plate.

In equation (41) the first term gives the increase in T due to the heat penetration from the free surface. The last term gives the influence on T of the heat flux from the hot plate. The equations show that $T(X, 0) = 0$ and $\Theta(X, 0) = 0$.

If the heat flux is constant (42, 43) the temperature and moisture potential fields become linear with X and independent of time in the limit $Fo \to \infty$. If the heat flux is described by (44) the temperature and moisture transfer potential will become uniform throughout the layer in the limit $Fo \to \infty$, the whole layer is in thermodynamic equilibrium with the surrounding air. The way in which an arbitrary heat flux variation with time can be handled is given in the Appendix B. One further sees that the moisture transfer potential distribution has a simple relation to the temperature distribution:

$$\Theta(X, Fo) = \frac{Lu\,Pn}{\nu}\,T(X, Fo) \qquad (47)$$

which follows directly from (4) and (5), with $Lu\,\partial^2\Theta/\partial X^2 = 0$ and the initial conditions (6). The combination $Lu\,Pn/\nu$ gives the lag of the moisture potential field, with respect to the temperature field.

The equations can be used for the heat penetration in dry material if one puts $\Lambda = \varphi = \nu = 1$.

To substantiate the obtained results a numerical example will be given now. It is needed of course to take into consideration the five points mentioned in the introduction to select a representative example. If the material has a fine porous structure with a partly colloidal nature, while no important shrinkage effects occur during drying, these conditions can be expected to be fulfilled. Condition 5 will certainly be satisfied in drying phenomena. Moreover the simplification of Makovozov must be acceptable. This can be done in either of two ways (i) the Posnov number is much higher than the Luikov number, or (ii) the beginning of the drying process is only considered where no large moisture content gradients have been built up. When $Lu < 1{\cdot}0$ the temperature gradients are already developed and thus form the main driving force for the mass flux. Finally the criterion φ must remain positive, which means that $(1 - \varepsilon)\,Ko\,Lu\,Pn < 1$. In [7] experimental values for a_m and δ are available for some materials (Luikov [7], p. 268, Table VII). Some values are given in Table 1.

Table 1

Material	Lu	δ $10^2(°C^{-1})$
1. Wood (fir), with 25% moisture [7]	0·019	2·0
2. Peat (200% moisture) [7]	0·40	2·4
3. Kaolin (10% moisture) [7]	0·05	0·11
4. Kaolin (47% moisture) [7]	0·73	0·19
5. Sawdust cakes [own measurements]	0·07	2·50

One can see that for the drying of wood, peat and the sawdust cakes (cakes formed by compressing sawdust with an agar–agar solution), and koalin with a low moisture content the Luikov number can be small in comparison with the Posnov number.

A value for δ of 0·01°C^{-1} can be considered representative. When a temperature difference of 60 degC (say) exists between the surrounding air and the initial temperature of the material and the moisture content in equilibrium with the surrounding air ($\Delta u_* = c_m(\theta_0 - \theta_*)$) is 1·0 kg/kg a value of 0·6 for Pn will result. A Bi_q value of 2·5 can be easily obtained in experimental situations with forced convection (e.g. $\alpha_q = 58{\cdot}15$ W/m^2°C, $\lambda_q = 0{\cdot}698$ W/m°C, $D = 0{\cdot}03$ m). If further $c_q = 12{\cdot}37$ J/kg°C and r is taken to be 2510·4 J/kg and $\phi_q = 139{\cdot}56$ W/m^2 then $Ki_q = 0{\cdot}9$ and $Ko = 5{\cdot}0$. If $(1 - \varepsilon)\,Lu < 0{\cdot}33$ the condition $\varphi > 0$ is fulfilled. The equations (42, 43, 45) and (46) where programmed on an IBM 1620 digital computer with the following set of variables:

$$Lu = 0{\cdot}4, 0{\cdot}02; \qquad \varepsilon = 0{\cdot}2, 0{\cdot}6, 0{\cdot}8, 1{\cdot}0;$$
$$Pn = 0{\cdot}6; \qquad Ki_q = 0{\cdot}9;$$
$$Ko = 5{\cdot}0; \qquad Bi_q = 2{\cdot}5.$$

The distributions where calculated at dimensionless times Fo = 0·05, 0·10, 0·40, 0·80, 1·60, 3·20, 6·40. In representing the results use has been made of the equation (47). Some typical results are given in the Figs. 2–4. In Fig. 2 (a, b, c) the temperature and moisture potential distributions are given for the constant heat flux situation with two values of ε (0·2 and 1·0). The distributions are both nonsymmetric. There is a maximum in the moisture content up to $Fo = 0{\cdot}20$, this maximum shifts towards the free surface during the drying process. The moisture content at the hot plate does not drop instantaneously to its final value as is the case when the temperature of the hot plate is constant. Values for Θ (in Fig. 2) which are higher than 1·0 can occur because the reference moisture

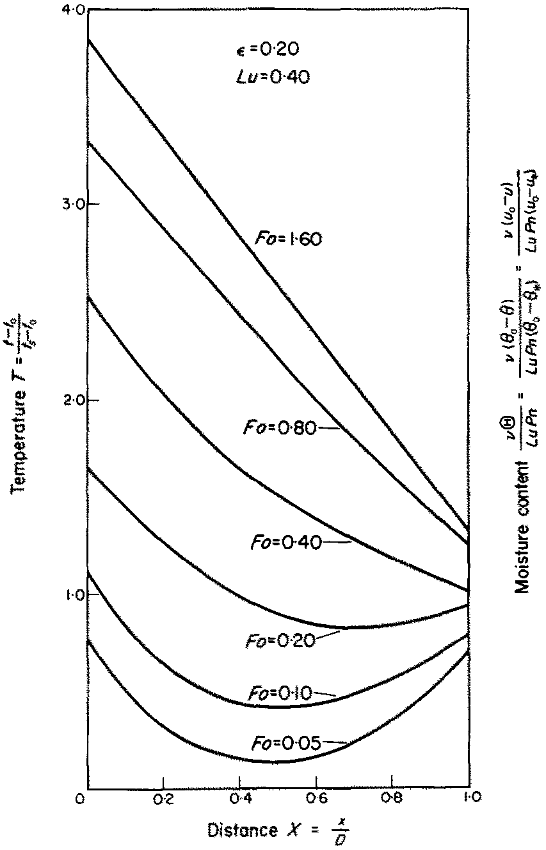

FIG. 2(a–c). Temperature and moisture transfer potential distributions during contact drying. (a) For $\varepsilon = 0{\cdot}2$.

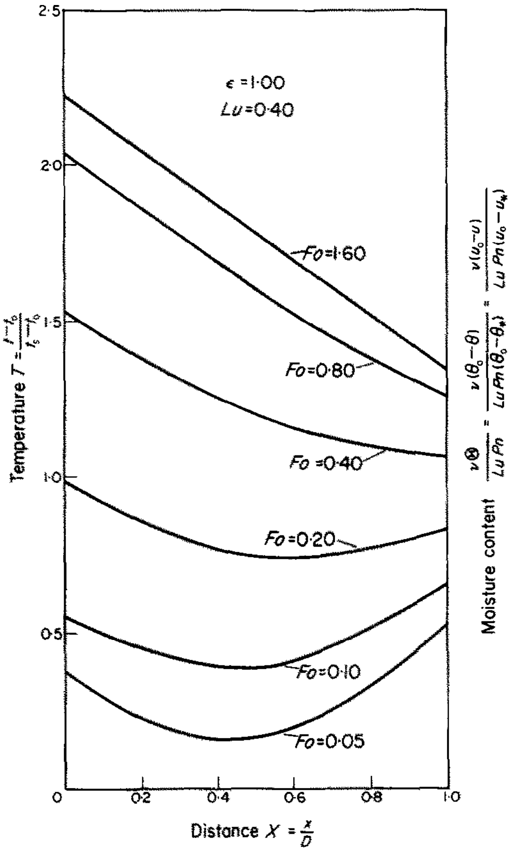

FIG. 2(b). For $\varepsilon = 1{\cdot}0$, $Lu = 0{\cdot}4$ in both cases.

content (θ_*) is the moisture content in equilibrium with the surrounding air whereas the temperature of the material can become higher than t_s so that θ may be smaller than θ_*.

In Fig. 3 temperature and moisture potential distributions are given for a heat flux which decreases exponentially with time. Two "relaxation times" $\sigma = 1{\cdot}0$ and $\sigma = 0{\cdot}5$ were used.

In Fig. 4 the influence of ε is given. A low value of ε gives higher temperatures in the material, because less heat is needed for evaporation of moisture.

Possibilities of comparison of the analytical results with experimental data in literature are very restricted because of the scarcity of such data.

In our own experiments with a decreasing heat flux qualitatively the same picture was found as the figures reflect. The moisture content showed a maximum (this corresponds with a minimum in the moisture transfer potential) which shifted towards the free surface in the course of time. The experimental curves however show an inflection point in the moisture potential distribution after some time ($Fo \sim 0{\cdot}3$). This is caused by the moisture content dependency of the coefficients in the constitutive equations.† Solution by finite difference methods of the differential equations would be necessary

† Luikov [15], showed that introduction of a hyperbolic type of diffusion equation can also account for the inflection point in the curves.

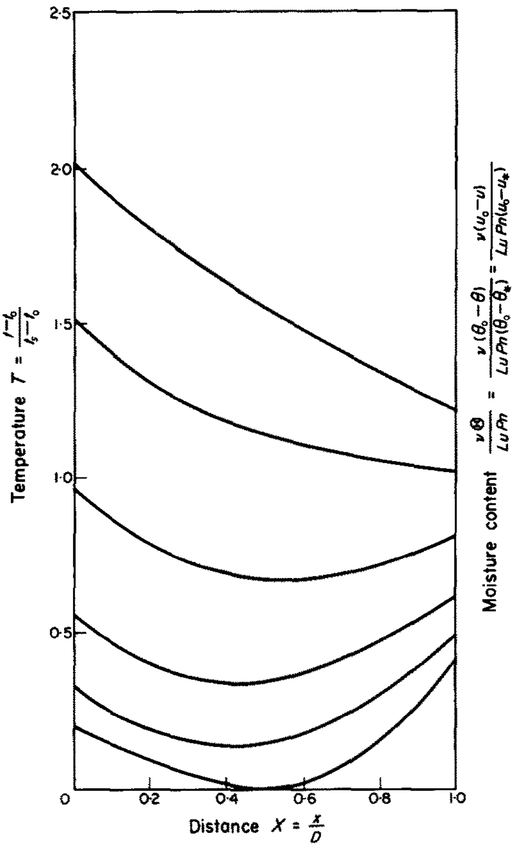

FIG. 2(c). For $\varepsilon = 0{\cdot}8$ and $Lu = 0{\cdot}02$; constant heat flux ($Ki_q = 0{\cdot}9$).

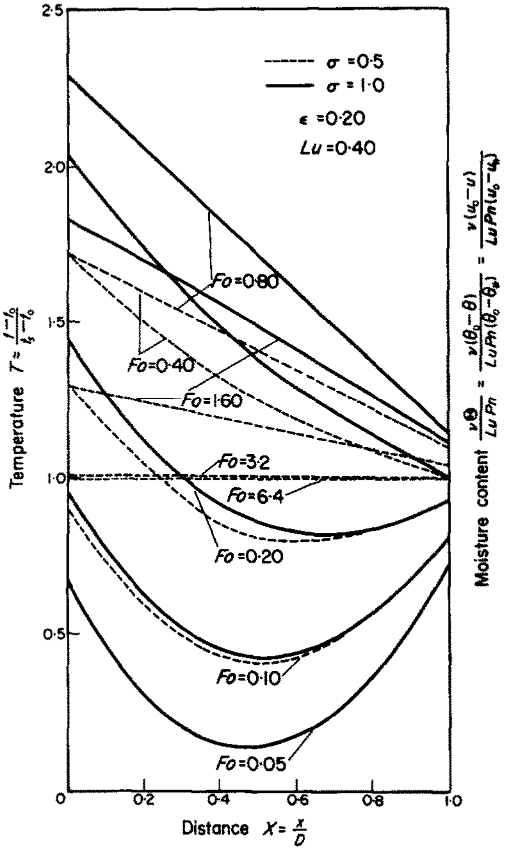

FIG. 3(a–b). Temperature and moisture transfer potential distributions during contact drying. (a) For $\varepsilon = 0{\cdot}20$ and $Lu = 0{\cdot}4$.

to predict the phenomenon of inflection points.

Englberger [14] studied moisture content and temperature distributions during combined convection and contact drying of kaolin. His results differ from our equations. The moisture content distributions began to show maxima shifting to the free surface, only after several hours of drying. In this case apparently the first term in equation (5) has a considerable influence during the first hours of drying, when the wide capillaries are still filled with liquid. From inspection of the concentration dependency of *Lu* and *Pn* in Table 1 this behaviour can be anticipated. *Lu* decreases sharply with decreasing moisture content, while δ remains of the same order of magnitude.

Finally the following remarks must be made. Although admittedly equations (38) and (41) give a highly simplified picture of the complex transport phenomena in contact drying, the equations can give sufficiently accurate results for engineering application. Especially if the calculation is done in stages with the right coefficients for each stage. Further studies on the numerical solution of the total set of equations (4) and (5) with variable coefficients is needed, to provide a more refined description of the drying process.

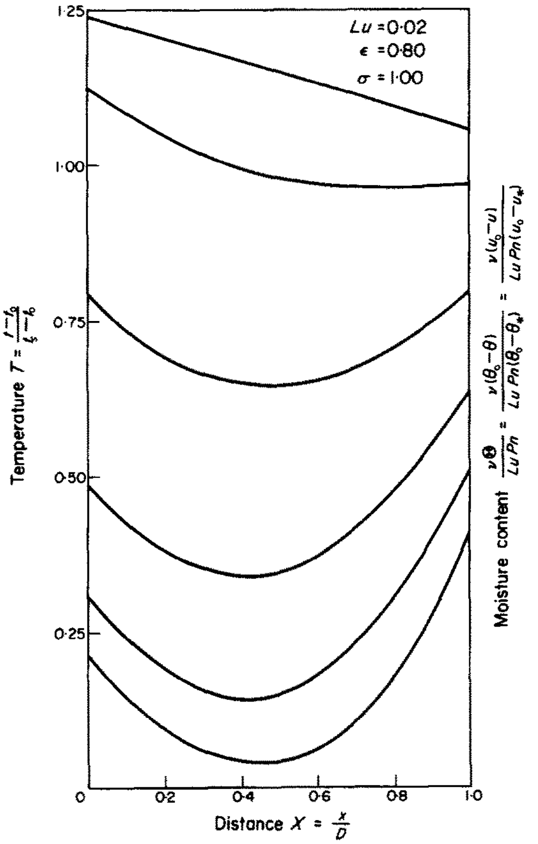

Fig. 3(b). For $\varepsilon = 0{\cdot}8$ and $Lu = 0{\cdot}02$; heat flux decreases exponentially with time ($Ki_q(Fo) = Ki_q^0 \exp - Fo/\sigma$).

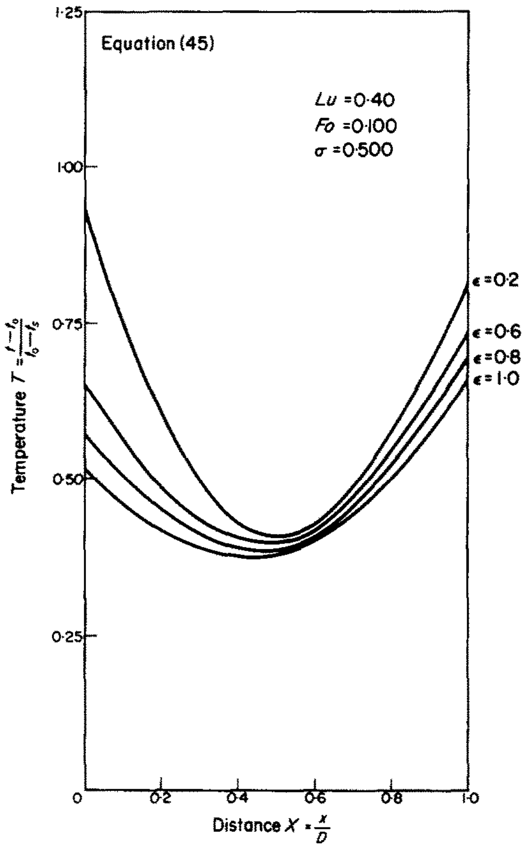

Fig. 4. The influence of the phase change criterion on the temperature and moisture transfer potential distribution during contact drying; heat flux decreases exponentially with time.

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APPENDIX A

The equation (37) can be written as:

$$\bar{T}(X,s) = \frac{\varphi\, Bi_q \cosh\left[X\sqrt{\left(\frac{s}{v}\right)}\right]}{s\left\{(1+\Pi)\sqrt{\left(\frac{s}{v}\right)}\,.\sinh\left[\sqrt{\left(\frac{s}{v}\right)}\right] + \varphi\, Bi_q \cosh\left[\sqrt{\left(\frac{s}{v}\right)}\right]\right\}}$$

$$+\overline{Ki}_q(s)\,\Lambda \frac{(1+\Pi)\cosh\left\{\sqrt{\left(\frac{s}{v}\right)}.(1-X)\right\} + \varphi\, Bi_q\sqrt{\left(\frac{v}{s}\right)}\sinh\left\{\sqrt{\left(\frac{s}{v}\right)}.(1-X)\right\}}{(1+\Pi)\sqrt{\left(\frac{s}{v}\right)}.\sinh\left[\sqrt{\left(\frac{s}{v}\right)}\right] + \varphi\, Bi_q\cosh\left[\sqrt{\left(\frac{s}{v}\right)}\right]}$$

$$= \frac{g(s)}{h(s)} + \overline{Ki}_q(s)\left\{\frac{n(s)}{m(s)}\right\}. \qquad \text{(A.1)}$$

The inversion of the first term in the right-hand side gives, by use of the Heaviside expansion theorem, Churchill [12], p. 169 etc.

$$\mathscr{L}^{-1}\left\{\frac{g(s)}{h(s)}\right\} = \lim_{s\to 0}\left\{\frac{g(s)}{h'(s)}\right\} + \sum_{n=1}^{\infty}\left\{\frac{g(s_n)}{h'(s_n)}\right\}\exp.(s_n Fo). \qquad \text{(A.2)}$$

The s_n are the poles of $h(s)$ except the zero; they lie all on the negative real axis and are given by the characteristic equation:

$$(1+\Pi)\sqrt{\left(\frac{s}{v}\right)}.\sin h\left[\sqrt{\left(\frac{s}{v}\right)}\right] + \varphi\, Bi_q\cosh\left[\sqrt{\left(\frac{s}{v}\right)}\right] = 0.$$

Changing to circular sines and cosines one obtains

$$\mu\tan\mu = \frac{\varphi\, Bi_q}{1+\Pi} = \Lambda\, Bi_q. \qquad \text{(A.3)}$$

Where μ_n are the roots:

$$\mu_n = i\sqrt{\left(\frac{s_n}{v}\right)}.$$

It is easy to show that

$$\lim_{s\to 0}\frac{g(s)}{h'(s)} = 1 \qquad \text{(A.4)}$$

and

$$\sum_{n=1}^{\infty}\frac{g(s_n)}{h^1(s_n)}\exp(s_n Fo) = -\sum_{n=1}^{\infty}\frac{2\sin\mu_n}{\mu_n+\sin\mu_n\cos\mu_n}\cos(\mu_n X)\exp.(-\mu_n^2 v Fo). \qquad \text{(A.5)}$$

The inversion of the second member of equation (A.1), right-hand side is found by the convolution theorem [12] p. 38:

$$\mathscr{L}^{-1}\left\{\overline{Ki}_q(s)\frac{n(s)}{m(s)}\right\} = \int_0^{Fo} Ki_q(\tilde{\tau})\left[\mathscr{L}^{-1}\left\{\frac{n(s)}{m(s)}\right\}\right]_{Fo-\tilde{\tau}} d\tilde{\tau}. \qquad \text{(A.6)}$$

The right-hand side can be evaluated with the inversion method, we obtain finally:

$$\mathscr{L}^{-1}\left\{\overline{Ki}_q(s)\frac{n(s)}{m(s)}\right\}$$

$$= \int_0^{Fo} Ki_q(\tilde{\tau})\, \nu A \sum_{n=1}^{\infty} \frac{2\mu_n}{\mu_n + \sin\mu_n \cos\mu_n}$$

$$\cos(\mu_n X)\exp\left[-\mu_n^2\nu(Fo-\tilde{\tau})\right]\mathrm{d}\tilde{\tau}. \qquad \text{(A.7)}$$

Equations (A.4, A.5, A.7) give the desired expression for $T(X, Fo)$ as given in equation (38). Provided the summation and integration in (A.7) can be interchanged. This will be allowed if the infinite series is uniform convergent and if the integral

$$\int_0^{Fo} Ki_q(\tilde{\tau})\exp\left[-\mu_n^2\nu(Fo-\tilde{\tau})\right]\mathrm{d}\tilde{\tau} \qquad \text{(A.8)}$$

exists for $0 \leqslant \tilde{\tau} \leqslant Fo$. The first condition will be fullfilled as can be seen by the ratio test; the quotient between two terms is smaller than one and becomes independent of $\tilde{\tau}$ in the limit for $n \to \infty$. The only restriction is the existence of the integral (A.8).

APPENDIX B

The integrals in equation (38) and (41) can be calculated by the following numerical procedure. Let $Ki_q(Fo)$ be known by experiment as a function of Fo (e.g. Fig. 5). By using Simpsons rule for numerical integration we obtain for the integral in (38) taking $2m$ intervals:

$$\int_0^{Fo} Ki_q(\tilde{\tau})\exp\left[-\mu_n^2\nu(Fo-\tilde{\tau})\right]\mathrm{d}\tilde{\tau}$$

$$= \frac{Fo}{6m}\Bigg\{Ki_q(0) + Ki_q(Fo) + 4\sum_{j=1,3,}^{2m-1} Ki_q(\tilde{\tau}_j)\exp\left[-\mu_n^2\nu(Fo-\tilde{\tau}_j)\right] + 2\sum_{j=2,4,}^{2m-2} Ki_q(\tilde{\tau}_j)\exp\left[-\mu_n^2\nu(Fo-\tilde{\tau}_j)\right]\Bigg\}$$

$$\equiv C(Fo, \mu_n). \qquad \text{(B.1)}$$

This integral will have to be calculated for each μ_n giving a number of $C(Fo, \mu_n)$. These can be

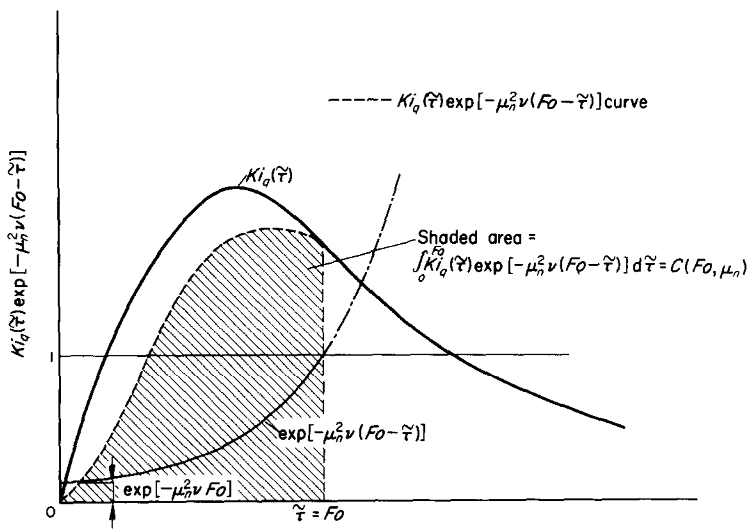

FIG. 5. Schematic procedure for calculation of $C(Fo, \mu_n)$.

used in the second summation of equation (38) and (41).

Although the method is tedious for desk computations, it allows the solution of problems where the heat flux to the drying material is a more or less arbitrary function of time.

Résumé—En employant des équations obtenues par Luikov [7–9] comme point de départ, on analyse le séchage d'une couche de matériau humide en contact avec une plaque chaude. Makovozov [5–6] a analysé un cas spécial, c'est-à-dire celui d'une température constante de la plaque chaude; nous nous attacherons spécialement (1) au cas d'un flux de chaleur constant et (2) au das d'un flux de chaleur diminuant exponentiellement avec le temps. Nous supposerons comme Makovozov que (1) l'influence du gradient de pression sur le mouvement de l'humidité est négligeable et (2) que le transport de l'humidité a lieu principalement sous l'effet des gradients de température. Nous avons essayé d'obtenir aussi une équation plus générale sans la restriction mentionnée en dernier lieu, mais nous avons reconcontré des difficultés insurmontables dues à la dissymétrie des conditions aux limites qui ont été supposées. L'influence des paramètres sans dimensions sur les distributions de température et de potentiel d'humidité est illustrée par des exemples numériques. Les résultats sont comparées avec les rares résultats expérimentaux de la littérature.

Zusammenfassung—Von der von Luikov [7–9] abgeleiteten Gleichung ausgehend, wurde die Trocknung einer Schicht feuchten Materials bei Berührung mit einer grossen Platte analysiert. Makovozov [5, 6] analysierte einen Spezialfall nämlich den konstanter Temperatur der Heizplatte; wir berücksichtigen (i) den Fall konstanten Wärmestroms un (ii) de Fall, mit der Zeit exponentiel abnehmenden Wärmestroms.

Wie Makovozov nehmen wir an (i), dass die Druckgradienteinflüsse auf die Feuchtigkeitsbewegung vernachlässigbar sind und (ii) dass der Feuchtigkeitstransport vorwiegend als Folge des Temperaturgradienten stattfindet. Wir bemühten uns auch eine allgemeinere Gleichung ohne die zuletzt erwähnten Einschränkungen abzuleiten; dem aber stellten sich unüberwindbare Schwierigkeiten entgegen wegen der Asymmetrie der Randbedingungen. Der Einfluss dimensionsloser Parameter auf Temperatur- und Feuchtigkeitsverteilung wird durch numerische Beispiele gezeigt. Die Ergebnisse werden mit den seltenen Versuchsdaten in der Literatur verglichen.

Аннотация—Используя уравнения, выведенные Лыковым [7–9], проведен анализ процесса сушки слоя влажного материала, находящегося в контакте с горячей пластиной. Маковозов [5, 6] рассматривал случай постоянной температуры на горячей пластине. Мы же рассматриваем случай постоянного теплового потока и случай, когда тепловой поток возрастает экспоненциально со временем. Как и Маковозов, мы полагаем, что (1) влияние градиента давления на перемещение влаги пренебрежимо мало и (2) что перенос влаги осуществляется в основном за счёт действия температурных градиентов. Мы попытались также получить общее уравнение, не прибегая к последнему ограничению, но встретились с величайшими трудностями из-за ассиметрии используемых граничных условий. Влияние безразмерных параметров на распределение потенциалов температуры и влаги проиллюстрировано численными примерами. Проведено сравнение результатов со скудными экспериментальными данными, имеющимся в литературе.